MOTION IIT-JEE
(Where Faith Counts the Success)

MATHEMATICS
TARGET IIT JEE

VECTOR

THEORY AND EXERCISE BOOKLET

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JEE Syllabus :

Addition of vectors, scalar multiplication, scalar products, dot and cross products, scalar triple products and their geometrical interpretations, direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

A. DEFINITIONS

A vector may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. A is called the initial point & B is called the terminal point. The magnitude of vector $\vec{AB}$ is expressed by $|\vec{AB}|$.

The modulus, or magnitude, of a vector is the positive number which is the measure of its length. The modulus of the vector $\vec{a}$ is sometimes denoted by $|\vec{a}|$, and sometimes by the corresponding symbol *a* in italics. The vector which has the same modulus as $\vec{a}$, but the opposite direction, is called the negative of $\vec{a}$, and is denoted by $-\vec{a}$.

Let the vectors $\vec{a}$, $\vec{b}$ be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then the inclination of the vectors, or the angle between them, is defined as that angle AOB which does not exceed π. Thus if θ denote this inclination, $0 \le \theta \le \pi$. When the inclination is $\pi/2$. The vectors are said to be perpendicular, when it is 0 or π they are parallel.

ZERO VECTOR : A vector of zero magnitude i.e. which has the same initial & terminal point, is called a Zero Vector. It is denoted by $\vec{0}$.

UNIT VECTOR : A vector of unit magnitude in direction of $\vec{a}$ vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

EQUAL VECTORS : Two vectors $\vec{a}$ & $\vec{b}$ are said to be equal if they have the same magnitude, direction & represent the same physical quantity. This is denoted symbolically by $\vec{a} = \vec{b}$.

COLLINEAR VECTORS : Two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **Parallel Vectors**. If they have the same direction they are named as like vectors otherwise unlike vectors.

Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

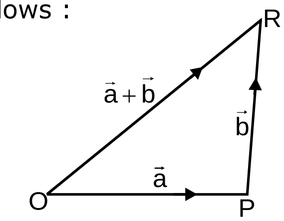
COPLANAR VECTORS : A given number of vectors are called coplanar if their directed line segments are all parallel to the same plane. Note that "Two Vectors Are Always Coplanar".

Remark : Vectors as defined above are usually called free vectors, since the value of such a vector depends only on its length and direction and is independent of its position in space. A single free vector cannot therefore completely represent the effect of a localized vector quantity, such as a force acting on a rigid body. This effect depends on the line of action of the force; and it will be shown later that two free vectors are necessary for its specification.

B. ADDITION AND SUBTRACTION OF VECTORS

(1) Triangle Law : The manner in which the vector quantities of mechanics and physics are compounded is expressed by the triangle law of addition, which may be stated as follows :

If three points O, P, R are chosen so that $\overrightarrow{OP} = \vec{a}$ and $\overrightarrow{PR} = \vec{b}$ then the vector $\overrightarrow{OR}$ is called the (vector) sum or resultant of $\vec{a}$ and $\vec{b}$. Denoting this resultant by $\vec{c}$, we write $\vec{c} = \vec{a} + \vec{b}$.



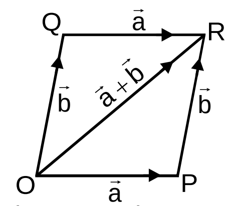
borrowing the sign + from algebra, and using the term vector addition for the process by which the resultant $\vec{c}$ is obtained from the component vectors $\vec{a}$ and $\vec{b}$. The above definition is not an arbitrary mathematical assumption. It is an expression of the way in which the vector quantities of physics and mechanics are compounded.

(2) Parallelogram Law : If the sum of two vectors $\vec{a} = \overrightarrow{OP}$ and $\vec{b} = \overrightarrow{OQ}$ is the vector $\overrightarrow{OR}$ determined by the diagonal of the parallelogram of which OP and OQ are sides.

The triangle law of addition is identical with the parallelogram law

Further since $\overrightarrow{QR} = \overrightarrow{OP} = \vec{a}$ it follows that $\vec{b} + \vec{a} = \overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR}$,

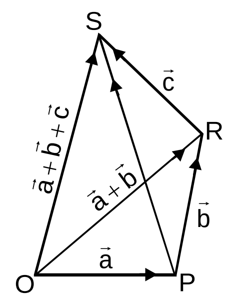
showing that $\vec{b} + \vec{a} = \vec{a} + \vec{b} = \vec{r}$ (say).



(3) Associative & Commutative Law : If we add another vector $\vec{c} = \overrightarrow{RS}$, obtaining the result

$\overrightarrow{OS} = \vec{r} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c} = \vec{c} + (\vec{a} + \vec{b})$.

But a glance at fig. shows that this vector is also $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$ = $\vec{a} + (\vec{b} + \vec{c}) = (\vec{b} + \vec{c}) + \vec{a}$, and the argument obviously holds for any number of vectors. Hence the commutative and associative laws hold for the addition of any number of vectors. The sum is independent of the order and the grouping of the terms.



We have already stated that $-\vec{b}$ is to be understood as the vector which has the same length as $\vec{b}$, but the opposite direction. The subtraction of $\vec{b}$ from $\vec{a}$ is to be understood as the addition of $-\vec{b}$ to $\vec{a}$. We denote this by $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$,

borrowing the – sign from algebra. Thus to subtract the vector $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ and add. $\vec{a} - \vec{b}$ is represented by the (undrawn) diagonal QP ; for $\vec{a} - \vec{b} = \overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{QP}$.

For the particular case in which $\vec{b} = \vec{a}$ we have $\vec{a} - \vec{a} = 0$.

All zero vectors are regarded as equal irrespective of direction. Indeed we may say that the direction of a zero vector is arbitrary. Vectors other than the zero vector are called proper vectors. For any vectors $\vec{a}$ and $\vec{b}$ we have the following inequality $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (the triangle inequality), geometrically expressing the fact that in a triangle the sum of its two sides is greater than the third side if the vectors are not parallel. This inequality is obviously valid for any number of vectors : $|\vec{a} + \vec{b} + \ldots\ldots\ldots + \vec{\ell}| \le |\vec{a}| + |\vec{b}| + \ldots\ldots\ldots + |\vec{\ell}|$.

Ex.1 ABCDE is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}, \overrightarrow{AE}, \overrightarrow{BC}, \overrightarrow{DC}, \overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

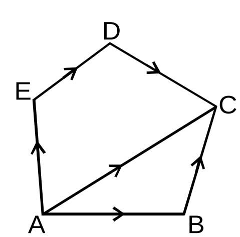
Sol. Let R be the resultant force.

$\therefore$ R = $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$\therefore$ R = $(\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$

= $\overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$

= $3\overrightarrow{AC}$. Hence proved.



(4) Multiplication of Vector by Scalars : If $\vec{a}$ is a vector & m is a scalar, then m$\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is | m | times that of $\vec{a}$. This multiplication is called SCALAR MULTIPLICATION. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$m(\vec{a}) = (\vec{a})m = m\vec{a}$ $\quad\quad$ $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$ $\quad\quad$ $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

(5) Position Vector : With an assigned point O as origin, the position of any point P is specified uniquely by the vector $\overrightarrow{OP}$, which is called the position vector of P relative to O. It will be found convenient to denote the position vectors of the points A, B, C, ...Z by the corresponding small Clarendon symbols $\vec{a}$, $\vec{b}$, $\vec{c}$, ...$\vec{z}$ With this notation the vector AB is $\vec{b} - \vec{a}$. For

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}.$$

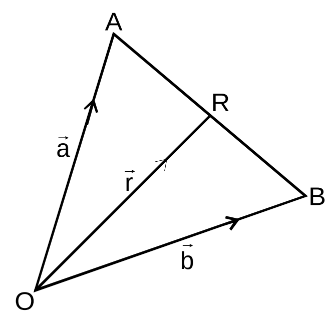
A point with position vector $\vec{r}$ is often referred to as the point $\vec{r}$.

C. SECTION FORMULA

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$. Note p.v. of mid point of AB = $\dfrac{\vec{a}+\vec{b}}{2}$.

Let A, B be the two points, and $\vec{a}$, $\vec{b}$ their position vectors relative to the origin O. Then the position vector of the point R which divides AB in the ratio m : n, may be found in terms of $\vec{a}$ and $\vec{b}$. For since $\overrightarrow{AR} = m\overrightarrow{RB}$, it follows that $n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$

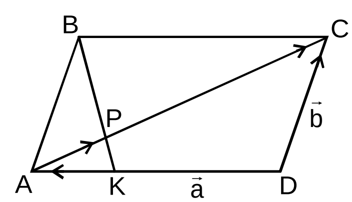
whence $\quad \vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}. \quad\quad (m + n \neq 0)$



This is the required expression for the position vector of R. The reasoning holds whether the ratio m : n is positive or negative. In the latter case R is outside the segment AB. If the ratio lies between 0 and −1, R is outside AB and nearer to A. If the ratio lies between −1 and −∞, R is outside AB and nearer to B.

For the particular case in which m = n, the above formula given $\frac{1}{2}(\vec{a} + \vec{b})$ for the position vector of the mid-point of AB.

Ex.2 The side AD of the parallelogram ABCD is divided into n equal parts and the first division point (point K) is joined to the vertex B (Fig.). Find the parts into which the diagonal AC is divided by the half-line BK.



Sol. Let $\overrightarrow{DC} = \vec{b}, \overrightarrow{DA} = \vec{a}$, and $\overrightarrow{AP} = \alpha\overrightarrow{AC}$. We express the vector $\overrightarrow{AP}$ in terms of the vectors $\vec{a}$ and $\vec{b}$ in two ways : (1) $\overrightarrow{AP} = \alpha\overrightarrow{AC} = \alpha(\vec{b} - \vec{a}) = \alpha\vec{b} - \alpha\vec{a}$; (2) $\overrightarrow{AP} = \overrightarrow{AK} + \overrightarrow{KP} = -\frac{1}{n}\vec{a} + \alpha\overrightarrow{KB}$

$= -\frac{1}{n}\vec{a} + \alpha(\frac{1}{n}\vec{a} + \vec{b}) = \frac{\alpha - 1}{n}\vec{a} + \alpha\vec{b}$ ($\overrightarrow{KP} = \alpha\overrightarrow{KB}$, since $\Delta APK \sim \Delta BPC$).

Since only one representation of a vector in terms of two noncollinear vectors is possible, we have : $\frac{\alpha - 1}{n} = -\alpha$, whence $\alpha = \frac{1}{n+1}$. This means that $\overrightarrow{AP} = \frac{1}{n+1}\overrightarrow{AC}$, and then, we see AP : PC = 1 : n.

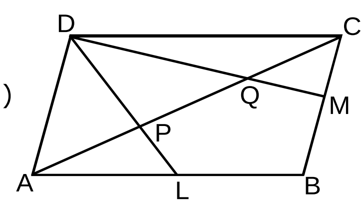
D. VECTOR EQUATION OF A LINE

Parametric vector equation of a line passing through two point A($\vec{a}$) & B($\vec{b}$) is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ where t is a parameter. If the line passes through the point A($\vec{a}$) & is parallel to the vector b then its equation is, $\vec{r} = \vec{a} + t\vec{b}$

If P is a point on this straight line the vector $\overrightarrow{AP}$ is parallel to $\vec{b}$, and is therefore equal to b, where t is some real number positive for points on one side of A, and negative for points on the other, varying form point to point. Thus, if a is the position vector of A, that of P is $\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + t\vec{b}$ ....(1)
And since any point on the given straight line has a position vector given by (1) for some value of t, we may speak of (1) as a vector equation of the straight line.
To find a vector equation of the straight line passing through the points A and B, whose position vectors are a and b, we observe that $\overrightarrow{AB} = \vec{b} - \vec{a}$; so that straight lines is one through the point A parallel to $\vec{b} - \vec{a}$. Its vector equation is therefore r = $\vec{a} + t(\vec{b} - \vec{a})$ or $\vec{r} = (1 - t)\vec{a} - t\vec{b}$ ......(3)
The three points A, B, P are collinear; and if the linear equation (3) connecting their position vectors is written $(1 - t)\vec{a} + t\vec{b} - \vec{r} = 0$,
with all the terms on one side, the algebraic sum of the co-efficients of the vectors is zero. This is the necessary and sufficient condition that three points should be collinear.

Ex.3 If ABCD is a parallelogram and L, M are the mid points of the sides AB and BC respectively, show that DL, AC meet at point of trisection and similarly DM and AC.

Sol. Let $\overrightarrow{AB} = \vec{a}, \overrightarrow{AD} = \vec{b}$ in the parallelogram ABCD. Then $\overrightarrow{AL} = \frac{\vec{a}}{2}, \overrightarrow{BM} = \frac{\vec{b}}{2}$



$\therefore$ let DL, DM meet AC in P and Q respectively $\overrightarrow{AP} = \lambda\overrightarrow{AC} = \lambda(\vec{a} + \vec{b})$ ....(1)

Also $\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} = \vec{b} + \mu\overrightarrow{DL} = \vec{b} + \mu(\overrightarrow{DA} + \overrightarrow{AL})$ i.e. $\overrightarrow{AP} = \vec{b} + \mu\left(-\vec{b} + \frac{\vec{a}}{2}\right)$ ....(2)

Equating the coefficients of $\vec{a}, \vec{b}$ in (1) and (2), we get $\lambda = \frac{\mu}{2}$, $\lambda = 1 - \mu \Rightarrow \lambda = \frac{1}{3}$

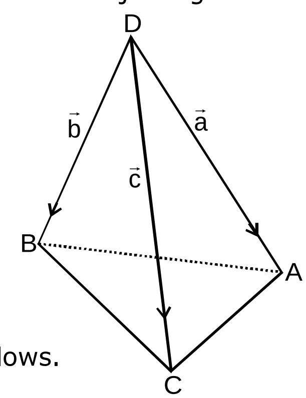
$\therefore$ $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AC}$ i.e. P is a point of trisection of AC. Similarly, we can prove $\overrightarrow{AQ} = \frac{2}{3}\overrightarrow{AC}$

Bisector of the angle between two straight lines. To find the equation of the bisector of the angle between the straight lines OA and OB, parallel to the unit vectors $\hat{a}$ and $\hat{b}$ respectively, take the point O as origin and let P be any point on the bisector. Then, if PN is drawn parallel to AO cutting OB in N, the angles OPN and NOP are equal, and ON = NP. But these are parallel to $\hat{b}$ and $\hat{a}$ respectively, so that $\overrightarrow{ON} = t\hat{b}$ and $\overrightarrow{NP} = t\hat{a}$, where t is some real number. The position vector of P is therefore $\vec{r} = t(\hat{a} + \hat{b})$. This is the required equation of the bisector, the value of t varying as P moves along the line. The bisector OP' of the supplementary angle B'OA is the bisector of the angle between straight lines whose directions are those of $\hat{a}$ and $-\hat{b}$; and its equation is therefore $\vec{r} = t(\hat{a} - \hat{b})$.

If $\vec{a}$ and $\vec{b}$ are not unit vectors, the equations of the above bisectors are $\vec{r} = t\left(\frac{\vec{a}}{a} \pm \frac{\vec{b}}{b}\right)$.

Ex.4 The lines joining the vertices of tetrahedron to the centroids of the opposite faces are concurrent.

Sol. Let ABCD be the tetrahedron. Take D as origin of position vectors. Then the line joining D to the centroid of the face ABC is $\vec{r} = s(\vec{a} + \vec{b} + \vec{c})$.



Also the centroid of the face DAC is the point $\frac{1}{3}(\vec{a} + \vec{c})$; and

the line joining this to B is $\vec{r} = t\vec{b} + (1 - t)\frac{\vec{a}+\vec{c}}{3}$.

These two lines intersect at the point for which s = t = $\frac{1}{4}$, that is

the point $\frac{1}{4}(\vec{a} + \vec{b} + \vec{c})$. From the symmetry of this result the theorem follows.

Non-parametric equation of straight line. Consider the straight line through A in the direction of the unit vector $\hat{e}$. For any point r on this line, the vector $\vec{r} - \vec{a}$ is parallel to $\hat{e}$, so that

$$(\vec{r} - \vec{a}) \times \hat{e} = 0 \qquad ....(i)$$

is one form of the equation of the line. The perpendicular distance from a point P($\vec{p}$) to the line has magnitude AP sin θ, which is $|(\vec{p} - \vec{a}) \times \hat{e}|$. It is thus the magnitude of the vector obtained by substituting $\vec{p}$ for $\vec{r}$ in the first member of equation (i). The position vector of N, the foot of the perpendiuclar, is $\vec{a} + \hat{e}.(\vec{p} - \vec{a})\hat{e}$. The vector $\overrightarrow{PN} = \overrightarrow{PA} + \overrightarrow{AN} = \vec{a} - \vec{p} + \hat{e}.(\vec{p} - \vec{a})\hat{e}$.

Ex.5 Line L_1 is parallel to vector $\vec{\alpha} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and passes through a point A(7, 6, 2) and line L_2 is parallel to a vector $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ and passes through a point B(5, 3, 4). Now a line L_3 parallel to a vector $\vec{r} = 2\hat{i} - 2\hat{j} - \hat{k}$ intersects the lines L_1 and L_2 at points C and D respectively, Find $|\overrightarrow{CD}|$.

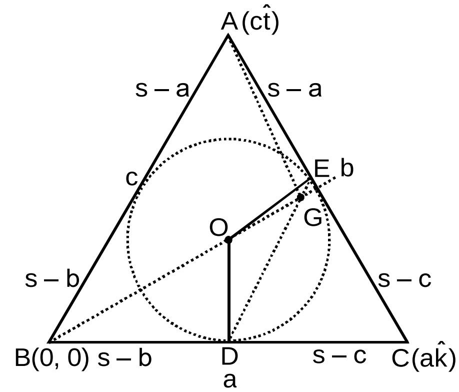
Sol. $\vec{r}_1 = 7\hat{i} + 6\hat{j} + 2\hat{k} + \lambda(-3\hat{i} + 2\hat{j} + 4\hat{k})$ $\vec{r}_2 = 5\hat{i} + 3\hat{j} + 4\hat{k} + \mu(2\hat{i} + \hat{j} + 3\hat{k})$

$\Rightarrow$ $\overrightarrow{CD} = (3\lambda + 2\mu - 2)\hat{i} + (-2\lambda + \mu - 3)\hat{j} + (-4\lambda + 3\mu + 2)\hat{k}, r = (2\hat{i} - 2\hat{j} - \hat{k})$

$\Rightarrow$ $\frac{3\lambda + 2\mu - 2}{2} = \frac{-2\lambda + \mu - 3}{-2} = \frac{4\lambda - 3\mu - 2}{1}$ $\Rightarrow$ λ = 2 and μ = 1 $\Rightarrow$ $\overrightarrow{CD} = 6\hat{i} - 6\hat{j} - 3\hat{k}$ $\Rightarrow$ $|\overrightarrow{CD}| = 9$

Ex.6 The in-circle of the triangle ABC touches its sides at D, E, F. If O is the centre of the incircle and BO meets DE at G, use vector method to prove that AG is perpendicular to BG.

Sol. Let B be taken as the initial point, Let the position vector of C and A be $a\hat{k}$ and $c\hat{t}$ respectively where $\hat{k}$ and $\hat{t}$ are unit vectors. With normal notations of ΔABC, the position vector of D is $(s-b)\hat{k}$ and that of E is $\dfrac{(s-c)c\hat{t}+(s-a)a\hat{k}}{b}$.



The equation of BO is $r = \lambda_1(\hat{i}+\hat{k})$ and that of DE is

$$r = (s-b)\hat{k} + \lambda_2\left[\frac{(s-c)c\hat{t}+(s-a)a\hat{k}}{b} - (s-b)\hat{k}\right]$$

These lines intersect at G, where

$$\lambda_1(\hat{t}+\hat{k}) = (s-b)\hat{k} + \lambda_2\left[\frac{(s-c)c\hat{t}+(s-a)a\hat{k}}{b} - (s-b)\hat{k}\right]$$

Equating the coefficients of $\hat{t}$ and $\hat{k}$, we get $\Rightarrow \lambda_1 = \dfrac{c(s-c)\lambda_2}{b}$ and $\lambda_1 = s - b + \dfrac{(s-a)a-(s-b)b}{b}\,\lambda_2$.

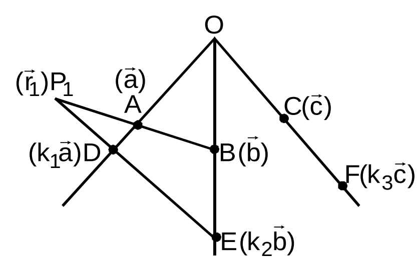
or $\lambda_1 = s - b + \dfrac{s(a-b)-(a^2-b^2)}{(s-c)c}\,\lambda_1 = s - b + \dfrac{(a-b)(s-a-b)}{(s-c)c}\,\lambda_1 \Rightarrow \lambda_1\left[1+\dfrac{a-b}{c}\right] = s - b \Rightarrow \lambda_1 = \dfrac{c}{2}$

Hence position vector of G is $\dfrac{c}{2}(\hat{t}+\hat{k})$

$\overrightarrow{AG} = \dfrac{c}{2}(\hat{t}+\hat{k}) - c\hat{t} = \dfrac{c}{2}(\hat{k}+\hat{t})$ which is perpendicular to BG i.e. $\dfrac{c}{2}(\hat{t}+\hat{k})$.

Ex.7 Three concurrent straight lines OA, OB & OC are produced D, E & F respectively. Use vectors to prove that the points of intersection of AB & DE ; BC & EF ; CA & FD are collinear.

Sol. AB : $\vec{r} = \vec{a} + \lambda(\vec{b}-\vec{a})$; DE : $\vec{r} = k_1\vec{a} + \mu(k_2\vec{b} - k_1\vec{a})$



For the point P_1, $(1-\lambda)\vec{a} + \lambda\vec{b} = (k_1 - k_1\mu)\vec{a} + \mu k_2\vec{b}$

$$\left.\begin{aligned} \therefore\quad \lambda &= \mu k_2 \\ \text{and } 1-\lambda &= k_1(1-\mu) \end{aligned}\right\}\quad \begin{aligned} &..........(1)\\ &..........(2)\end{aligned}$$

(1) & (2) gives $\mu = \dfrac{1-k_1}{k_2-k_1}$; $\lambda = \dfrac{(1-k_1)k_2}{k_2-k_1}$

$$\therefore\ \vec{r}_1 = \vec{a}\left[1-\frac{(1-k_1)k_2}{k_2-k_1}\right] + \frac{(1-k_1)k_2}{k_2-k_1}\vec{b} \quad \text{or} \quad \vec{r}_1 = \frac{k_1(k_2-1)}{k_2-k_1}\vec{a} - \frac{k_2(k_1-1)}{k_2-k_1}\vec{b}$$

or $\dfrac{(k_2-k_1)}{(k_1-1)(k_2-1)}\ \vec{r}_1 = \left(\dfrac{k_1}{k_1-1}\right)\vec{a} - \left(\dfrac{k_2}{k_2-1}\right)\vec{b}$ ....(3)

Similarly, $\dfrac{k_3-k_2}{(k_2-1)(k_3-1)}\ \vec{r}_2 = \left(\dfrac{k_2}{k_2-1}\right)\vec{b} - \left(\dfrac{k_3}{k_3-1}\right)\vec{c}$ ....(4)

and $\frac{k_1-k_3}{(k_3-1)(k_1-1)}\ \vec{r}_3=\left(\frac{k_3}{k_3-1}\right)\vec{c}-\left(\frac{k_1}{k_1-1}\right)\vec{a}$ ....(5)

($\vec{r}_2$ & $\vec{r}_3$ are the position vectors of P_2 & P_3 which are not shown)

(3) + (4) + (5) gives $\frac{(k_2-k_1)}{(k_1-1)(k_2-1)}\vec{r}_1+\frac{(k_3-k_2)}{(k_2-1)(k_3-1)}\vec{r}_2+\frac{(k_1-k_3)}{(k_3-1)(k_1-1)}\vec{r}_3=0$ [Note : $x\vec{r}_1+y\vec{r}_2+z\vec{r}_3=0$]

Also sum of the co-efficient of $\vec{r}_1,\vec{r}_2$ and $\vec{r}_3$ is $\frac{(k_3-1)(k_2-k_1)+(k_1-1)(k_3-k_2)+(k_2-1)(k_1-k_3)}{(k_1-1)(k_2-1)(k_3-1)}$

which is equal to zero. Hence, $\vec{r}_1,\vec{r}_2$ and $\vec{r}_3$ are collinear

E. SCALAR PRODUCT OF TWO VECTORS

Scalar quantities are of frequent occurrence, which depend each upon two vector quantities in such a way as to be jointly proportional to their magnitudes and to the cosine of their mutual inclination. An example of such is the work done by a force during a displacement of the body acted upon. We therefore find it convenient to adopt the following.

Definition. The scalar product of two vectors $\vec{a}$ and $\vec{b}$, whose directions are inclined at an angle θ, is the real number an cos θ, and is written $\vec{a}\cdot\vec{b}$ = ab cos θ = $\vec{b}\cdot\vec{a}$

The order of the factors may be reversed without altering the value of the product. Further, b cos θ is the resolute of $\vec{b}$ in the direction of $\vec{a}$, and a cos θ is the resolute of $\vec{a}$ in the direction of $\vec{b}$, positive or negative according as θ is acute or obtuse. Hence the scalar product of two vectors is the product of the modulus of either vector and the resolute of the other in its direction.

If two vectors $\vec{a}$, $\vec{b}$ are perpendicular, cos θ = 0, and their scalar product is zero. Hence the condition of perpendicularity of two proper vectors is expressed by $\vec{a}\cdot\vec{b}$ = 0

If the vectors have the same direction, cos θ = 1, and $\vec{a}\cdot\vec{b}$ = ab. If their directions are opposite, cos θ = –1, and $\vec{a}\cdot\vec{b}$ = –ab. The scalar product of any two unit vectors is equal to the cosine of the angle between their directions. When the factors are equal vectors their scalar product $\vec{a}\cdot\vec{a}$ is called the square of a, and is written a^2. Thus a^2 = $\vec{a}\cdot\vec{a}$ = a^2, the square of a vector being thus equal to the square of its modulus. The square of any unit vector is unity. In particular, $\hat{i}^2=\hat{j}^2=\hat{k}^2=1$,

but since these vectors are mutually perpendicular $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$.

These relations will be constantly employed.

If either factor is multiplied by a number, the scalar product is multiplied by these number.

for $(n\vec{a})\cdot\vec{b}$ = nab cos θ = $\vec{a}\cdot(n\vec{b})$.

since the scalar product is a number, it may occur as the numerical coefficient of $\vec{a}$ vector.

Thus $(\vec{a}\cdot\vec{b})\vec{c}$ is a vector, obtained on multiplying $\vec{c}$ by the number $\vec{a}\cdot\vec{b}$.

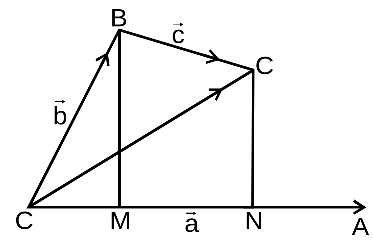
The combination $(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})$ of four vectors is simply the product of the two numbers $\vec{a}\cdot\vec{b}$ and $\vec{c}\cdot\vec{d}$.

Suppose that a vector $\vec{r}$ is resolved into two component vectors, one in the direction of the unity vector $\hat{e}$, and the other perpendicular to $\hat{e}$. The first of these, being the projection of $\vec{r}$ on $\vec{e}$, is $(\vec{r}\cdot\vec{e})\vec{e}$. The component vector perpendicular to $\vec{e}$ is therefore $\vec{r}-(\vec{r}\cdot\vec{e})\vec{e}$.

Similarly the rectangular resolution of Art. 7 may be expressed $\vec{r}=(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

Distributive Law : It is easy to show that the distributive law of multiplication holds for scalar products; that is, $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

The resolute of $\vec{b} + \vec{c}$ in the direction of a is the sum of the resolutes of $\vec{b}$ and $\vec{c}$ in the same direction. Consequently on multiplying each of these resolutes by $\vec{a}$, we have the required result. Fig. is drawn for the case in which all the resolutes are positive; but the argument holds also when one or more of them are negative.



Using the result of the preceding Art. we may write

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot (\overrightarrow{OC}) = \vec{a} \cdot ON = \vec{a}\ (OM + MN) = \vec{a} \cdot OM + \vec{a} \cdot MN = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$$

Repeated application of this result showns that the scalar product of two sums of vectors may be expanded as in ordinary algebra. Thus

$$(\vec{a} + \vec{b} + \ldots\ldots..).\ (1 + m + \ldots\ldots\ldots.) = \vec{a} \cdot 1 + \vec{a} \cdot m + \ldots\ldots.. + \vec{b} \cdot 1 + \vec{b} \cdot m + \ldots\ldots.$$

In particular, $(\vec{a} + \vec{b})^2 = a^2 + 2\vec{a} \cdot \vec{b} + b^2$, while $(\vec{a} + \vec{b}).\ (\vec{a} - \vec{b}) = a^2 - b^2$.

From the distributive law we may deduce a very useful formula for the scalar product of two vectors in terms of their rectangular components. For, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$$\vec{a} \cdot \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).\ (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = a_1b_1 + a_2b_2 + a_3b_3$$

since the vectors $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular. Thus the scalar product of two vectors is equal to the sum of the product of their corresponding rectangular components.

In particular, the square of a vector is equal to the sum of squares of its rectangular components. Also since the above scalar product is ab cos θ, the inclination of the vectors is given by

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}}$$

Direction Cosines : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then angles which this vector makes with the +ve directions OX, OY & OZ are called Direction Angles & their cosines are called the **Direction Cosines**.

$\cos\alpha = \dfrac{a_1}{|\vec{a}|}, \cos\beta = \dfrac{a_2}{|\vec{a}|}, \cos\gamma = \dfrac{a_3}{|\vec{a}|}$. Note that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Ex.8 Given $\vec{a} = 3\hat{i} + 2\hat{j} + 4\hat{k}\,;\vec{b} = 2\,(\hat{i} + \hat{k})$ and $\vec{c} = 4\hat{i} + 2\hat{j} + 3\hat{k}$. For what values of '$\alpha$' the equation, $x\vec{a} + y\vec{b} + z\vec{c} = \alpha\,(x\hat{i} + y\hat{j} + z\hat{k})$ has a non trivial solution.

Sol. Equating the components, $3x + 2y + 4z = \alpha x\ ;\ 2x + 2z = \alpha y\ \&\ 4x + 2y + 3z = \alpha z$

for non trivial solution $\begin{vmatrix} 3-\alpha & 2 & 4 \\ 2 & -\alpha & 2 \\ 4 & 2 & -\alpha \end{vmatrix} = 0$ (Use : $C_1 \to C_1 - C_2$) $\Rightarrow (\alpha + 1)^2\,(\alpha - 8) = 0 \Rightarrow \alpha = -1$ or 8

Ex.9 Prove cosine formula $c^2 = a^2 + b^2 - 2ab\cos C$ in triangle ABC.

Sol. If $\vec{a}, \vec{b}, \vec{c}$ are the vectors $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ we have $-\vec{c} = \vec{a} + \vec{b}$. On 'squaring' both sides of this equation we find $c^2 = a^2 + b^2 + 2a \cdot b = a^2 + b^2 - 2ab\cos C$, Since the inclination of $\vec{a}$ and $\vec{b}$ is $\pi - C$.